Lifted Inference for Relational Continuous Models

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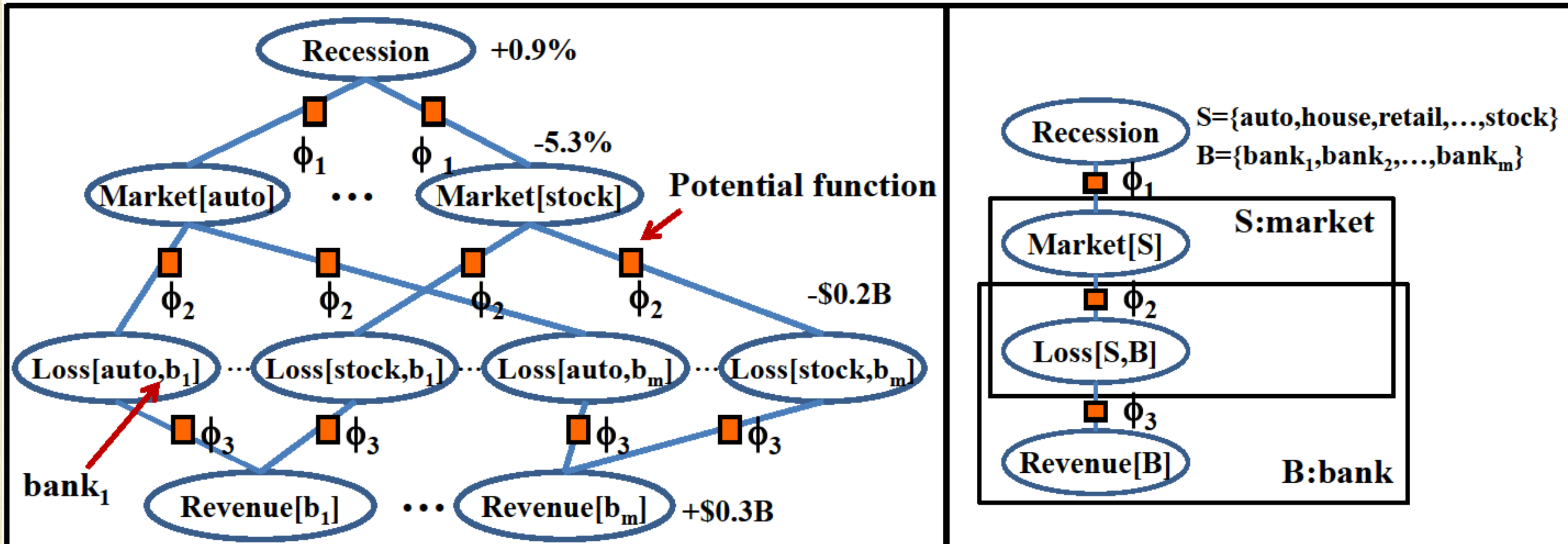
What are Relational Continuous Models (RCMs)?

- An example RCM



Recession: Recession of a country in [0,1] (i.e +0.9%)

Revenue[B]: Revenue of a bank(B) (i.e +$0.3B)

Loss[S,B]: Loss of a bank(B) in a sector(S) (i.e -$0.2B)

Market[S]: market index of a sector(S) (i.e -5.3%)

Why we use relational models?

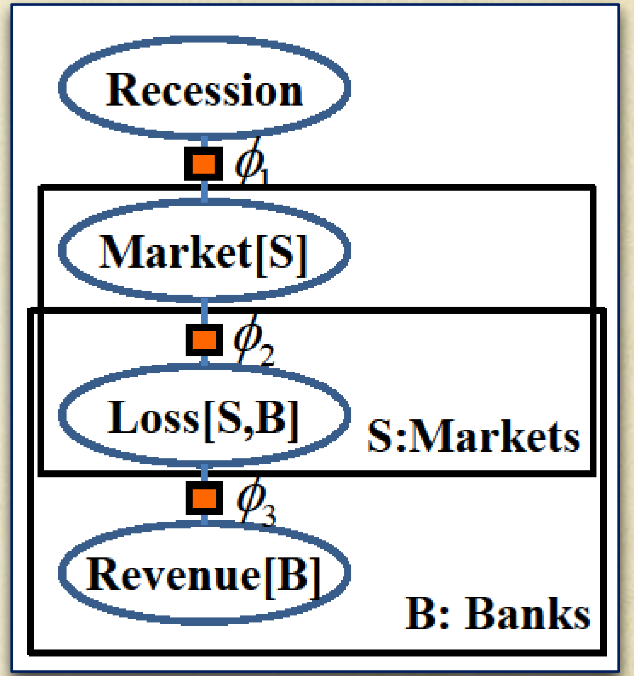
They allow compact representations and efficient inference possible (in some cases).

Contents

- Lifted Inference for Relational Continuous Models
  - Notations: Relational Continuous Models
  - Problem definition: an inference problem
  - Our lifted inference algorithm for Pair-wise Relational Normals
  - Complexity Analysis
  - Experimental Results

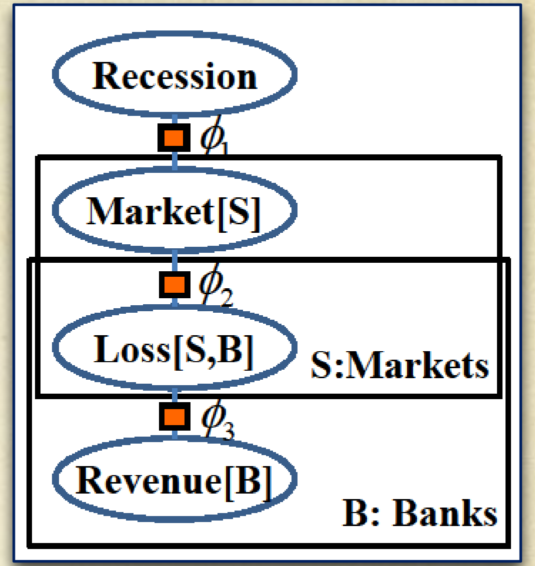
Notations: Relational Continuous Models (RCMs)

- Notations in RCMs
  - A set of parfactors
    - A parfactor g: (L,C,R, ϕ)
      - L: a set of logical variables (e.g. {S:Market,B:Banks})
      - C: constraints (e.g. Market ≠ stock)
      - R: a set of relational atoms (e.g. Market[S])
      - ϕ: a potential (e.g. ϕ_1(Recession, Market[S]))
  - There are three parfactors
    - ({S}, {}, {Recession, Market[S]}, ϕ_1)
    - ({S,B}, {}, {Market[S], Loss[S,B]}, ϕ_2)
    - ({B}, {}, {Loss[S,B], Revenue[B]}, ϕ_3)
  - Probability density of an instance



$$\prod_{s \in S} \phi_1(\mathrm{Recession}, Market(s)) \cdot \prod_{s \in S, b \in B} \phi_2(Market(s), Loss(s,b)) \cdot \prod_{s \in S, b \in B} \phi_3(Loss(s,b), \mathrm{Revenue}(b))$$

Problem Definition: an inference problem



- Given
  - A RCM
  - Observations

    (e.g. Market(auto) = -5.3%, Revenue(Pacific Bank)=-$0.2B)
  - Query random variables

    (e.g. Recession)
- Output
  - The marginal probability density of the query variables.
- Conditions
  - The potentials in the RCM are arity 2 with the following forms

$$\phi_{RN}(X[\ldots],Y[\ldots]) = c \cdot \exp\left(-\frac{(X[\ldots]-Y[\ldots])^2}{\sigma^2} \right)$$

  - There is an observation or a Gaussian prior for at least a random variable.

Relational Normal Potentials

- Pair-wise Relational Normals (RNs)
  - A family of Gaussian potentials with following forms
  - The potential is defined over two relational atoms (X[…], Y[…])

$$\phi_{RN}(X[\ldots],Y[\ldots]) = c \cdot \exp(-(X[\ldots]-Y[\ldots])^2/\sigma^2)$$

$$\phi_{RN}(Market[S], Loss[S,B]) = c \cdot \exp(-(Market[S]-Loss[S,B])^2/\sigma^2)$$

- Pair-wise Relational Normals with Means (RNMs)

$$\phi_{RNM}(X[\ldots],Y[\ldots]) = c \cdot \exp(-(X[\ldots]-Y[\ldots]-\mu)^2/\sigma^2)$$

Eliminating random variables in Gaussian Potentials

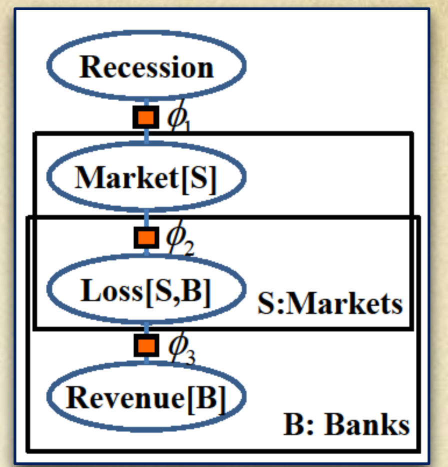
- Gaussian Potentials are quadratic exponential family.
  - For all random variables, it is a quadratic form (e.g. $R(b_i)$).

$$\exp(-aR(b_i)^2 + bR(b_i) + c)$$

  - a is a positive constant. b and c are terms without $R(b_i)$.
- Marginalizing with respect to a random variables, $R(b_i)$

$$\int_{R(b_i)} \exp(-aR(b_i)^2 + bR(b_i) + c) = \sqrt{\frac{\pi}{a}} \exp\left(\frac{b^2}{4a} + c\right)$$

Inversion Elimination



- Eliminating Loss[S,B]

Inversion Elimination (Eliminate Loss[S,B])

ϕ_2 : Market[S], Loss[S,B]
ϕ_3 : Loss[S,B], Revenue[B]

⇒ ϕ_4 : Market[S], Revenue[B]

Potential function — Variables

$$\int_{Loss[S,B]} \phi_2(Market[S], Loss[S,B]) \cdot \phi_3(Loss[S,B], Revenue[B])$$

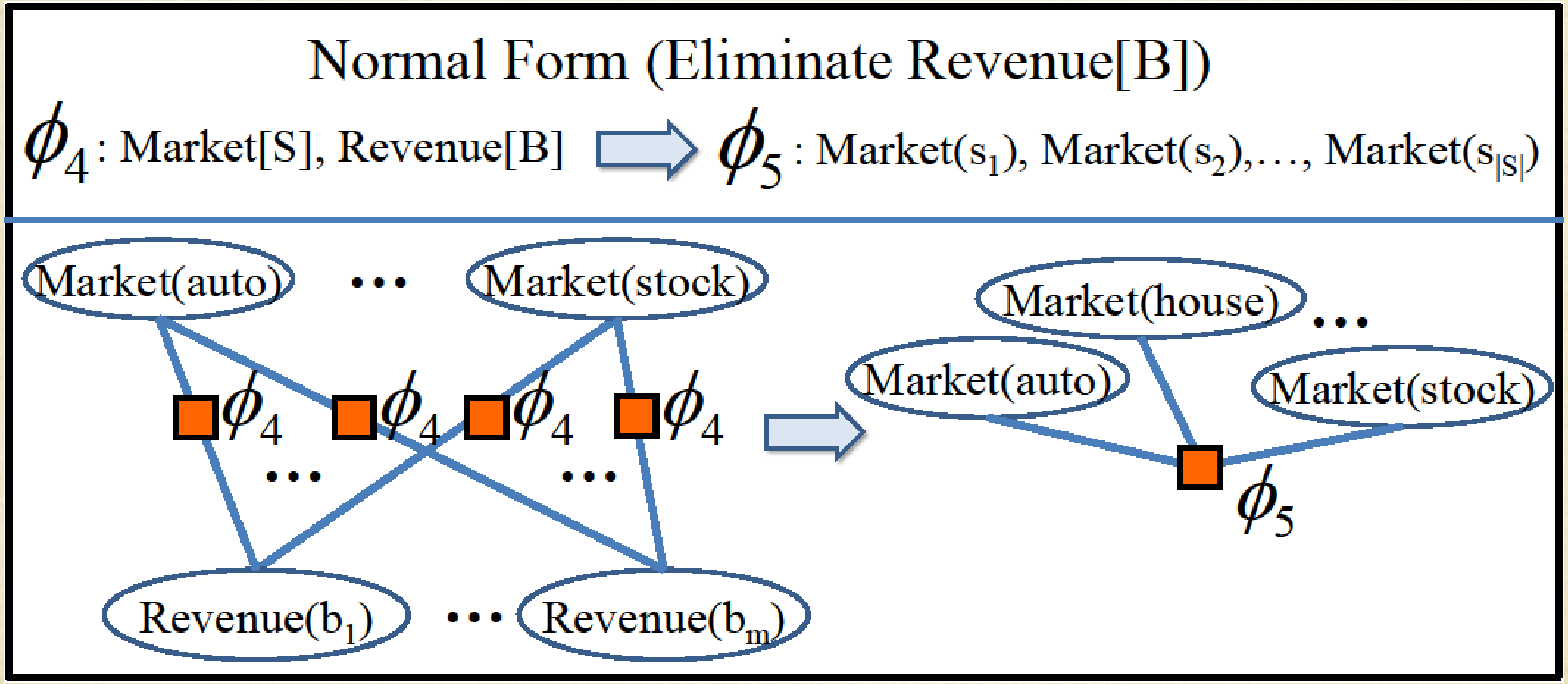
$$= c \cdot \int_{Loss[S,B]} \exp(-(Market[S] - Loss[S,B])^2/\sigma_1^2 - (Loss[S,B] - Revenue[B])^2/\sigma_2^2)$$

$$= c' \cdot \exp\left(-(Market[S] - Revenue[B])^2/(\sigma_1^2 + \sigma_2^2)\right)$$

$$= \phi_4\left(Market[S], Revenue[B]\right)$$

Eliminating random variables in a ground level

- Eliminating Revenue(b_1), ..., Revenue(b_m)



Eliminating random variables in a ground level

- Details of marginalization

$$\int_{R(b_i)} \prod_{s \in S} \phi_4(M(s), R(b_i)) dR(b_i) = \int_{R(b_i)} \exp\left(\sum_{s \in S} -\frac{(R(b_i) - M(s))^2}{\sigma^2} \right)$$

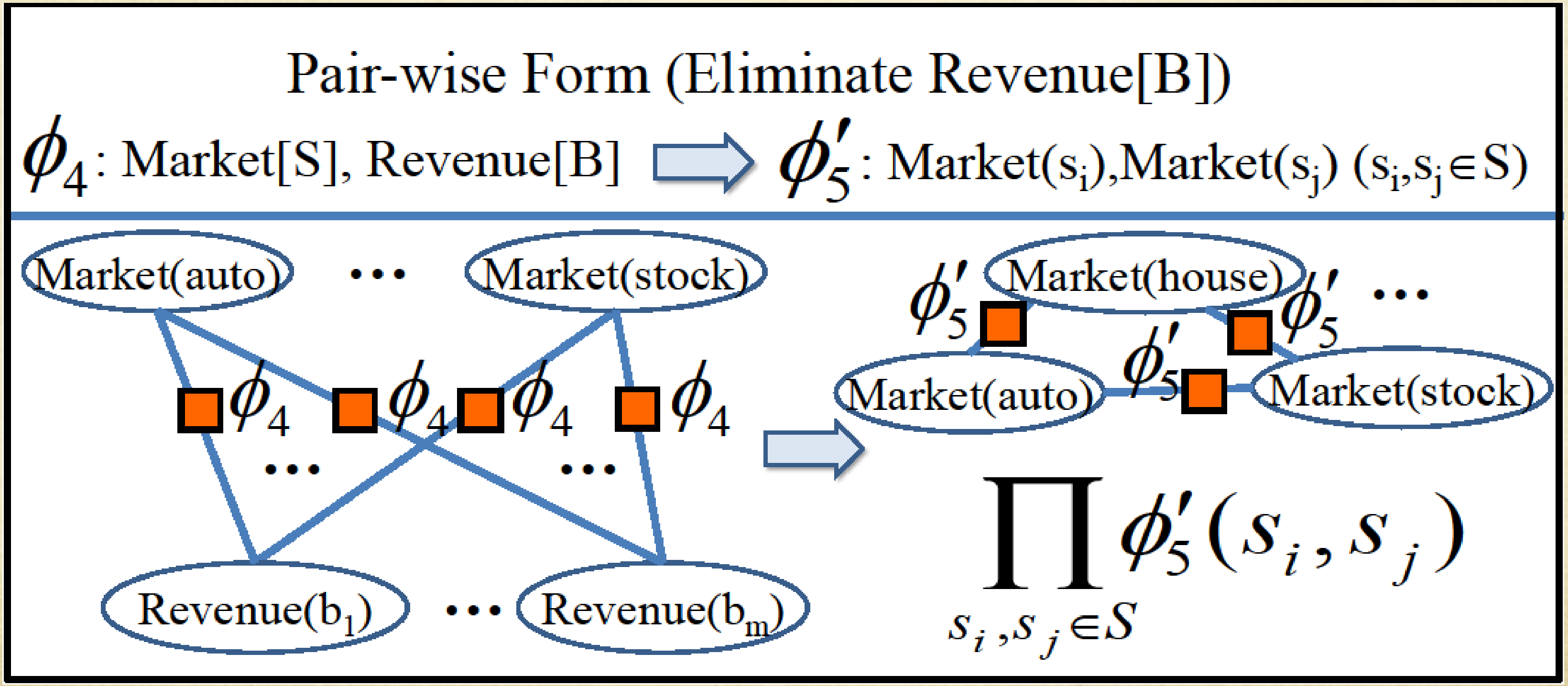
$$= \int_{R(b_i)} -\frac{|S|}{\sigma^2} R(b_i)^2 + \frac{2\left(\sum_{s \in S} M(s)\right)}{\sigma^2} R(b_i) - \frac{\left(\sum_{s \in S} M(s)^2\right)}{\sigma^2}$$

$$= c \cdot \exp\left(\frac{\left(\sum_{s \in S} M(s)\right)^2}{2\sigma^2 \cdot |S|} - \frac{\sum_{s \in S} M(s)^2}{2\sigma^2} \right)$$

$$= \phi_5\left(M(s_1), M(s_2), \cdots, M(s_{|S|})\right)$$

Our method – Eliminating random variables in a lifted level with the Pair-wise form

- We can make the pair-wise form after eliminating a random variable in the Pair-wise Relational Normals.



Our method – Eliminating random variables in a lifted level with the Pair-wise form

- Details of marginalization

$$\int_{R(b_i)} \prod_{s\in S} \phi_4(M(s),R(b_i))dR(b_i) = \int_{R(b_i)} \exp\left(\sum_{s\in S} -\frac{(R(b_i)-M(s))^2}{\sigma^2}\right)$$

$$= \int_{R(b_i)} -\frac{|S|}{\sigma^2}R(b_i)^2 + \frac{2\left(\sum_{s\in S} M(s)\right)}{\sigma^2}R(b_i) - \frac{\left(\sum_{s\in S} M(s)^2\right)}{\sigma^2}$$

$$= c\cdot \exp\left(\frac{\left(\sum_{s\in S} M(s)\right)^2}{2\sigma^2\cdot |S|} - \frac{\sum_{s\in S} M(s)^2}{2\sigma^2}\right)$$

$$= c\cdot \prod_{1\le i< j\le |S|} \exp\left(-\frac{(M(s_i)-M(s_j))^2}{2\sigma^2\cdot |S|}\right)$$

$$= c\cdot \prod_{1\le i< j\le |S|} \phi_5'(M(s_i),M(s_j))$$

Our method – Eliminating random variables in a lifted level with the Pair-wise form

- Generalization
  - We made some algebraic notations

$$X_{[m]} = \sum_{1 \le i \le m} x_i \qquad \text{e.g.} \quad M_{[|S|]} = \sum_{s \in S} M(s)$$

$$X_{[m]^2} = \sum_{1 \le i \le m} x_i^2 \qquad \text{e.g.} \quad M_{[|S|]^2} = \sum_{s \in S} M(s)^2$$

$$X_{[m][m]} = \sum_{1 \le i \le m} x_i \cdot x_j$$

$$\left(X_{[m]^2}\right)^2 = X_{[m]^2} + 2X_{[m][m]}$$

Computational Complexity

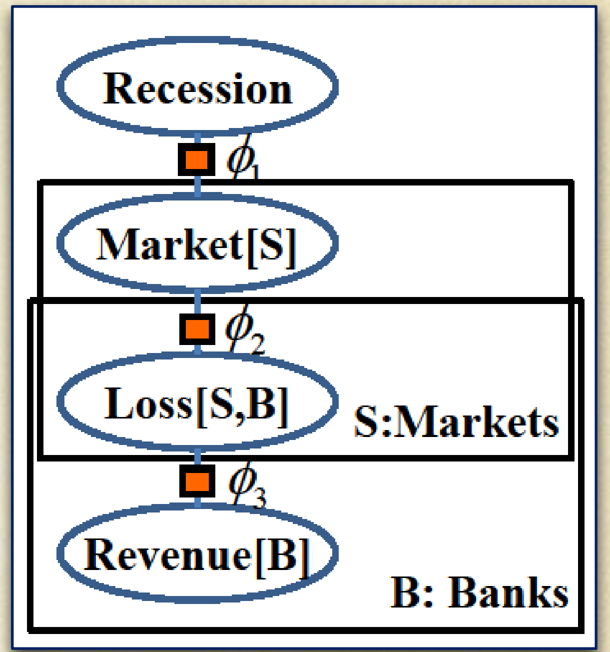
- Given
  - |U| is the all random variables
  - |N| is the relational atoms
  - Normally, |U|>>|N|
- Marginalization of n random variables in a ground level

$$O(n\cdot|U|^2)$$

- Marginalization of n random variables with our algorithm

$$O(n\cdot|N|^2)$$

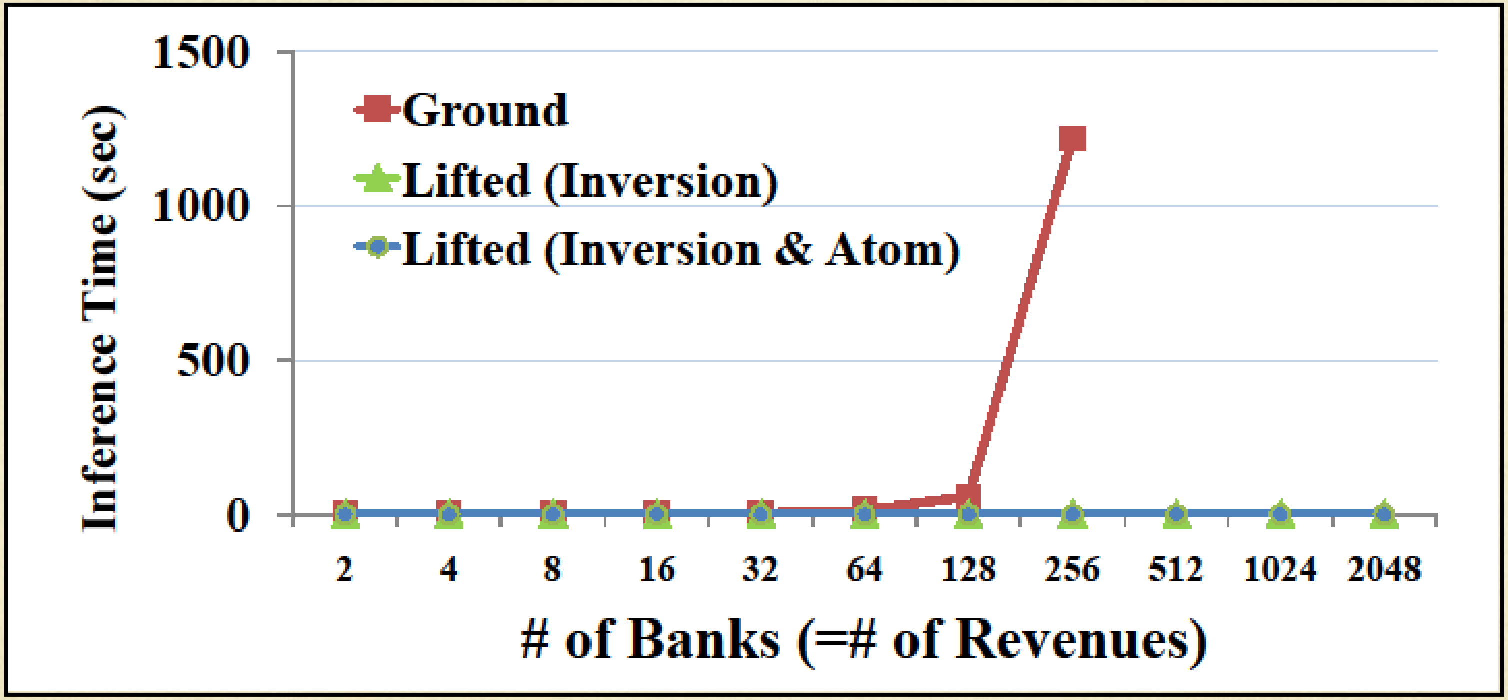
Experiments



- Experimental Settings
  - Given the econometric example
  - One observation for a market variable
  - One observation for a revenue variable
  - Output: Recession variable
  - Compare three algorithms
    - A ground inference
    - A lifted inference with only inversion
    - A lifted inference with inversion and pair-wise
- Exp 1: Increase the number of banks (|B|)
- Exp 2: Increase the number of market sectors (|S|)

Increase the number of Banks

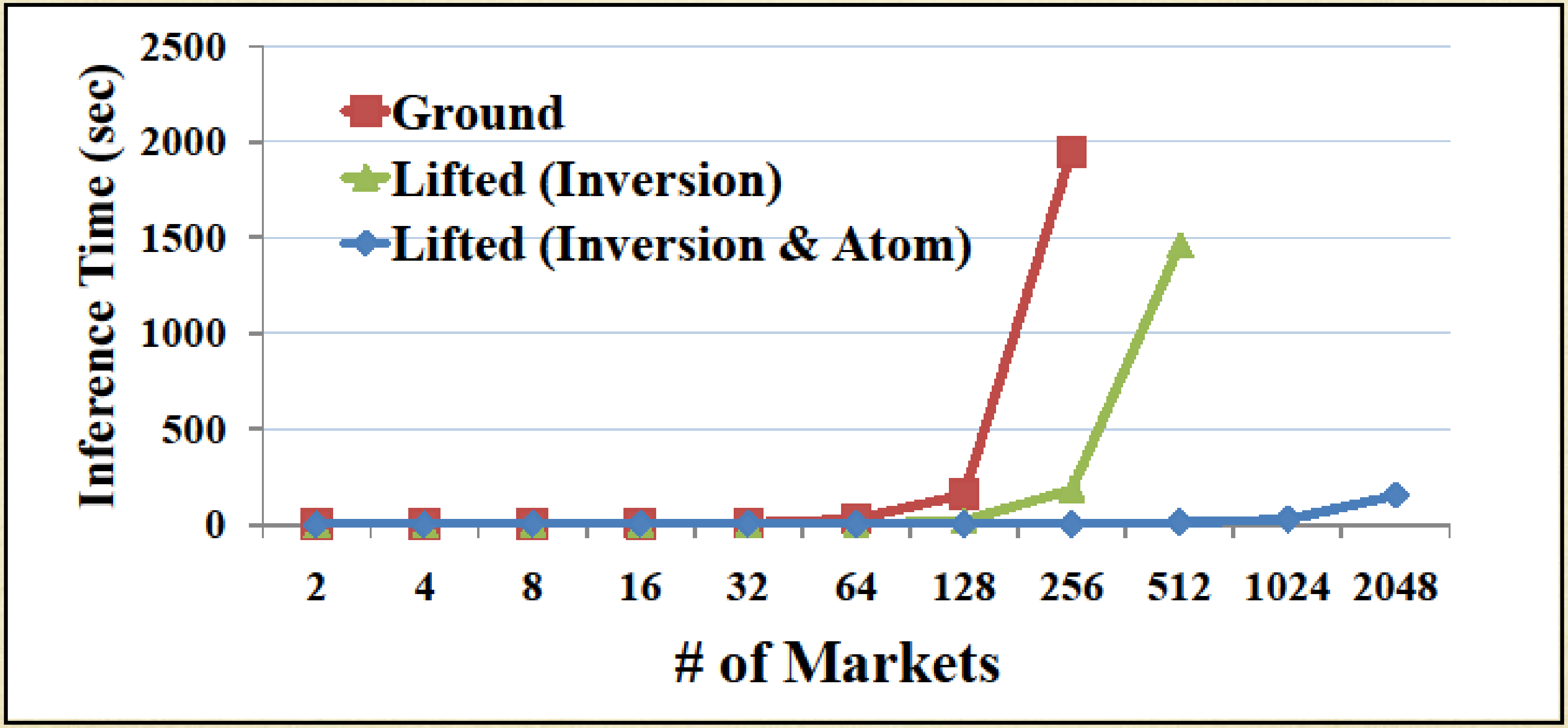
- Time for exact inference



- When # of banks is 2n, # of random variables is 4n+|Market|+1.
- The computational complexity of ground algorithm is quadratic $16n^3$.

Increase the number of Markets

- Time for exact inference



Additional Details in the Paper

- The paper includes
  - A condition which makes the relational normal is a probability density function.
  - A proof that the product of potentials is a probability density function given the condition.
  - Three conditions for exact lifted inference for general potentials (including non-Gaussian)
    - Analytically integrable
    - Closed under product operations
    - Closed under marginalization, thus represented with the product of relational pairwise form

Conclusion and Future work

- What we have seen so far
  - A linear time exact inference algorithm for continuous domains with Gaussian potentials.
  - The lifted algorithm substantially reduces the inference time compared to the ground-based algorithm.

- Future work
  - An algorithm for other potential (Gamma, Beta, Dirichlet ...) would be promising.
  - There are many applications with Hybrid domains.