Remember FOL signatures?

- What’s a signature?
Remember FOL signatures?

- What’s a signature?
  - constant, predicate and function symbols
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- What’s a signature?
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  - arity
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- What’s a signature?
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Remember FOL signatures?

- What’s a signature?
  - constant, predicate and function symbols
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    - constants: zero
Remember FOL signatures?

- What’s a signature?
  - constant, predicate and function symbols
  - arity
  - example
    - constants: `zero`
    - predicates: `greater(2), even?(1), isNat?(1)`
Remember FOL signatures?

What’s a signature?
- constant, predicate and function symbols
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- example
  - constants: *zero*
  - predicates: *greater(2)*, *even?(1)*, *isNat?(1)*
  - functions: *succ(1)*, *plus(2)*
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- Terms
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  - succ(plus(x, succ(succ(zero)))))
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- Terms
  - \(\text{succ}(\text{plus}(x, \text{succ}(\text{succ}(\text{zero}))))\)
  - \(\text{succ}(x, y)\)
Remember FOL signatures?

- What’s a signature?
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    - constants: `zero`
    - predicates: `greater(2)`, `even?(1)`, `isNat?(1)`
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- Terms
  - `succ(plus(x, succ(succ(zero))))`
  - `succ(x, y)`
  - `plus(zero, even?(x))`
Remember WFF’s?

- Base case: atoms
Remember WFF’s?

- Base case: atoms
  - `even?(succ(zero))`
Remember WFF’s?

- **Base case: atoms**
  - $\text{even?}(\text{succ}(\text{zero}))$
  - $\neg \text{plus}(x, \text{succ}(y)) = \text{zero}$
Remember WFF’s?

- Base case: atoms
  - `even?(succ(zero))`
  - `¬plus(x, succ(y)) = zero`
- General WFF’s
Remember WFF’s?

- Base case: atoms
  - even?(succ(zero))
  - ¬plus(x, succ(y)) = zero

- General WFF’s
  - ∀x even?(zero) → succ(succ(zero))
Remember WFF’s?

- Base case: atoms
  - \( \text{even?(succ(zero))} \)
  - \( \neg \text{plus}(x, succ(y)) = \text{zero} \)

- General WFF’s
  - \( \forall x \ \text{even?(zero)} \rightarrow \text{succ}(\text{succ}(\text{zero})) \)
  - \( \exists x \ \text{isNat?}(x) \land \forall y \ \text{isNat?}(y) \rightarrow \neg x = \text{succ}(y) \)
Remember WFF’s?

- **Base case: atoms**
  - even?(succ(zero))
  - ¬plus(x, succ(y)) = zero

- **General WFF’s**
  - ∀x even?(zero) → succ(succ(zero))
  - ∃x isNat?(x) ∧ ∀y isNat?(y) → ¬x = succ(y)
    - is ∀y isNat?(y) → ¬x = succ(y) closed?
Remember WFF’s?

- Base case: atoms
  - \(\text{even?}(\text{succ}(\text{zero}))\)
  - \(\neg\text{plus}(x, \text{succ}(y)) = \text{zero}\)

- General WFF’s
  - \(\forall x \; \text{even?}(\text{zero}) \rightarrow \text{succ}(\text{succ}(\text{zero}))\)
  - \(\exists x \; \text{isNat?}(x) \land \forall y \; \text{isNat?}(y) \rightarrow \neg x = \text{succ}(y)\)
    - is \(\forall y \; \text{isNat?}(y) \rightarrow \neg x = \text{succ}(y)\) closed?

- every even natural number except zero can be written as the sum of two natural numbers that are not even
We will assign meaning to our symbols and formulas
In this lecture - FOL semantics

We will assign meaning to our symbols and formulas
  - fix the domain of our problem - *universe*
We will assign meaning to our symbols and formulas

- fix the domain of our problem - *universe*
- interpret the symbols in this universe
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We will assign meaning to our symbols and formulas

- fix the domain of our problem - *universe*
- interpret the symbols in this universe
- assign meaning to variables
- evaluate formulas according to the interpretation and assignment
- relate formulas to each other according to how they evaluate in the semantics
Lecture Outline

- FOL models
  - What are FOL models?
  - Isomorphisms of models
- WFF satisfaction
  - Assignments
  - Interpreting the truth of WFF’s
  - Consequence and equivalence
Lecture Outline

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FOL Model Example

- Remember the signature at the beginning of the lecture
FOL Model Example

- Remember the signature at the beginning of the lecture
- One possible interpretation:

  Suppose we are talking about natural numbers
  - `zero` is 0
  - `greater-than` is `>`, so `{(1, 0), (2, 0), (2, 1), ...}
  - `even` is `{0, 2, 4, ...}
  - `isNat` is `N`
  - `succ` is a function that adds one to every number
  - `plus` is the sum function
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One possible interpretation:
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Another FOL Model Example

Same signature, another possible interpretation:
Another FOL Model Example

Same signature, another possible interpretation:

- Talking about natural numbers again
Another FOL Model Example

Same signature, another possible interpretation:

- Talking about natural numbers again
- \textit{isNat}? is \{1, 2, \ldots\}
Another FOL Model Example

Same signature, another possible interpretation:

- Talking about natural numbers again
- \textit{isNat}? is \{1, 2, ..\}
- \textit{zero} is 0
Another FOL Model Example

Same signature, another possible interpretation:

- Talking about natural numbers again
- \textit{isNat}? is \{1, 2, \ldots\}
- \textit{zero} is 0
- \textit{even}? is \{0, 2, 4, \ldots\}
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- \textit{isNat}? is \{1, 2, \ldots\}
- \textit{zero} is 0
- \textit{even}? is \{0, 2, 4, \ldots\}
- \textit{greater} is \(\geq\), so \{(0, 0), (1, 0), (1, 1), \ldots\}
Another FOL Model Example

Same signature, another possible interpretation:

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- `even?` is `{0, 2, 4, ...}`
- `greater` is `>=`, so `{(0, 0), (1, 0), (1, 1), ...}`
- `succ` is a function that adds two to every number
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- This time we are talking about the cars in a parking lot
Another FOL Model Example

Same signature, another possible interpretation:

- This time we are talking about the cars in a parking lot
- zero is the car in the center
Another FOL Model Example

Same signature, another possible interpretation:

- This time we are talking about the cars in a parking lot
- \textit{zero} is the car in the center
- \textit{greater} is such that a car is greater than another if its size is bigger
Another FOL Model Example

Same signature, another possible interpretation:

- This time we are talking about the cars in a parking lot
- zero is the car in the center
- greater is such that a car is greater than another if its size is bigger
- even? is the set of red cars
Another FOL Model Example

Same signature, another possible interpretation:

- This time we are talking about the cars in a parking lot
- `zero` is the car in the center
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- *zero* is the car in the center
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- *succ* is a function that returns, for each car, the nearest other car, or itself if there is no other car
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- \textit{plus} is a function that returns the bigger of the two cars
Given signature \((C, (F_k^k)_{k \geq 0}, (R^k_k)_{k \geq 0})\), a structure (interpretation, model) \(I\) consists of:
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- a function of arity \(k\) \(I(f)\) or \(f^I\) for each function symbol \(f \in F^k\) - \(I(f) : |I|^k \to |I|\)
Lecture Outline

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  - What are FOL models?
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An Example

- Consider a signature with constant symbol $\text{zero}$ and function symbol $\text{succ}$ (of arity 1)

Let $I$ be a structure where:

- $|I|$ is the set of natural numbers
- $\text{zero}_I = 0$
- $\text{succ}_I(x) = x + 1$ for all natural numbers $x$

Let $J$ be another structure with:

- the same universe and interpretation for $\text{zero}$
- $\text{succ}_J(x) = x + 2$ for all natural numbers $x$

What can you say about structures $I$ and $J$? Is there any way you can differentiate them with what you have?
An Example

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  [Additional details about $J$ could be here]
An Example

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  - $zero^I = 0$
  - $succ^I(x) = x + 1$ for all natural numbers $x$
- Let $J$ be another structure with:
  - the same universe and interpretation for zero
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Consider a signature with constant symbol zero and function symbol succ (of arity 1)

Let I be a structure where:
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An Example

- Consider a signature with constant symbol \textit{zero} and function symbol \textit{succ} (of arity 1)
- Let \textit{I} be a structure where:
  - \(|I|\) is the set of natural numbers
  - \(zero^I = 0\)
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- Let \textit{J} be another structure with:
  - the same universe and interpretation for \textit{zero}
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- What can you say about structures \textit{I} and \textit{J}?
Consider a signature with constant symbol zero and function symbol succ (of arity 1).

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Let $J$ be another structure with:
- the same universe and interpretation for zero
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What can you say about structures $I$ and $J$?

Is there any way you can differentiate them with what you have?
Another Example

- Consider a signature with constant symbol zero, function symbol \textit{succ} (of arity 1) and predicate symbol \textit{even} (of arity 1)
Consider a signature with constant symbol \textit{zero}, function symbol \textit{succ} (of arity 1) and predicate symbol \textit{even?} (of arity 1).

Let $I$, $J$ be structures like $I$, $J$ from the previous example, where additionally each interpret \textit{even?} as the set of even numbers.
Consider a signature with constant symbol \textit{zero}, function symbol \textit{succ} (of arity 1) and predicate symbol \textit{even}\(?)\ (of arity 1)

Let \(I, J\) be structures like \(I, J\) from the previous example, where additionally each interpret \textit{even}\(?)\ as the set of even numbers

What can you say about \(I\) and \(J\) now?
Isomorphic Models

- Given signature \((C, (F^k)_{k \geq 0}, (R^k)_{k \geq 0})\) and two structures \(I, J\):
Isomorphic Models

Given signature \((C, (F^k)_{k \geq 0}, (R^k)_{k \geq 0})\) and two structures \(I, J\):

- an isomorphism \(h : I \to J\) is a function \(h : |I| \to |J|\), such that:
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  - for each \(k \geq 0\), \(\rho \in R^k\) and \(k\) objects \(o_1, \ldots, o_k \in |I|\),
    \(\rho^I(o_1, \ldots, o_k)\) iff \(\rho^J(h(o_1), \ldots, h(o_k))\)
Given signature \((C, (F^k)_{k \geq 0}, (R^k)_{k \geq 0})\) and two structures \(I, J\):

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- for each \(k \geq 0\), \(f \in F^k\) and \(k\) objects \(o_1, ..., o_k \in |I|\), \(h(f^I(o_1, ..., o_k)) = f^J(h(o_1), ..., h(o_k))\)
Isomorphic Models

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    - for each \(k \geq 0\), \(f \in F^k\) and \(k\) objects \(o_1, ..., o_k \in |I|\), \(h(f^I(o_1, ..., o_k)) = f^J(h(o_1), ..., h(o_k))\)
  - Two structures \(I, J\) are isomorphic iff there is an isomorphism \(h : I \rightarrow J\)
Examples and Discussion

- What is the isomorphism in the case of the first example?
Examples and Discussion

- What is the isomorphism in the case of the first example?
- Can you think of a model that is isomorphic to $I$ in the second case? What is the isomorphism?
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- How about this signature: constant $\text{first}$, binary relation $\text{next}$
  - Structure $I$ with $|I| = \{0, 1, 2\}$ interprets $\text{first}$ as 0 and $\text{next}$ as $\{(0, 1), (1, 2)\}$
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  - Structure $I$ with $|I| = \{0, 1, 2\}$ interprets $first$ as 0 and $next$ as $\{(0, 1), (1, 2)\}$
  - Structure $J$ with $|J| = \{0, 1, 2, 3\}$ interprets $first$ as 0 and $next$ as $\{(0, 1), (1, 3)\}$
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  - Structure $M$ with $|M| = \{0, 1, 2, 3\}$ interprets $\text{first}$ as 0 and $\text{next}$ as $\{(0, 1), (1, 2), (2, 3)\}$

Which structures are isomorphic? What is the isomorphism?
How many structures isomorphic to $I$ can you think of?
How many isomorphic models are there?
Examples and Discussion

- What is the isomorphism in the case of the first example?
- Can you think of a model that is isomorphic to \( I \) in the second case? What is the isomorphism?
- How about this signature: constant \( \text{first} \), binary relation \( \text{next} \)
  - Structure \( I \) with \( |I| = \{0, 1, 2\} \) interprets \( \text{first} \) as 0 and \( \text{next} \) as \( \{(0, 1), (1, 2)\} \)
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  - Structure \( M \) with \( |M| = \{0, 1, 2, 3\} \) interprets \( \text{first} \) as 0 and \( \text{next} \) as \( \{(0, 1), (1, 2), (2, 3)\} \)
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Examples and Discussion

- What is the isomorphism in the case of the first example?
- Can you think of a model that is isomorphic to $I$ in the second case? What is the isomorphism?
- How about this signature: constant $\text{first}$, binary relation $\text{next}$
  - Structure $I$ with $|I| = \{0, 1, 2\}$ interprets $\text{first}$ as 0 and $\text{next}$ as $\{(0, 1), (1, 2)\}$
  - Structure $J$ with $|J| = \{0, 1, 2, 3\}$ interprets $\text{first}$ as 0 and $\text{next}$ as $\{(0, 1), (1, 3)\}$
  - Structure $M$ with $|M| = \{0, 1, 2, 3\}$ interprets $\text{first}$ as 0 and $\text{next}$ as $\{(0, 1), (1, 2), (2, 3)\}$
  - Which structures are isomorphic? What is the isomorphism?
  - How many structures isomorphic to $I$ can you think of?
- How many isomorphic models are there?
Lecture Outline

- FOL models
  - What are FOL models?
  - Isomorphisms of models
- WFF satisfaction
  - Assignments
  - Interpreting the truth of WFF’s
  - Consequence and equivalence
Assignments

- What about variables?
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Variable assignments $\alpha : \mathcal{V} \to |I|$ are extended to term assignments:

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Variable assignments $\alpha : V \rightarrow |I|$ are extended to term assignments:

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- if $f \in F^k$ is a function symbol of arity $k \geq 0$ and $t_1, \ldots, t_k$ are terms,
- then $f(t_1, \ldots, t_k)[\alpha] = f^I(t_1[\alpha], \ldots, t_k[\alpha])$
Consider a signature with constant symbol \textit{zero} and function symbols \textit{succ}(1) and \textit{plus}(2).
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Consider interpretation I with natural numbers as the universe and

- zero^I = 0
Example

- Consider a signature with constant symbol zero and function symbols succ(1) and plus(2)
- Consider interpretation $I$ with natural numbers as the universe and
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Example

- Consider a signature with constant symbol \( \text{zero} \) and function symbols \( \text{succ}(1) \) and \( \text{plus}(2) \)
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What are the interpretations of the following terms, considering $I$ and $\alpha$?

- $succ\left( plus\left( succ\left( succ\left( zero\right)\right),\ succ\left( x + y\right)\right)\right)$
Example

- Consider a signature with constant symbol `zero` and function symbols `succ(1)` and `plus(2)`
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- Consider `\mathcal{V} = \{x, y, z\}` assignment `\alpha` with
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The Satisfaction Relation

Now we can interpret the truth of a WFF $\phi$ with the respect to a model $I$ and an assignment $\alpha : \mathcal{V} \rightarrow |I|$.
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Satisfaction relation - we say that $\phi$ satisfies model $I$ and assignment $\alpha$ and write: $I, \alpha \models \phi$, or $I \models \phi[\alpha]$, or $\models_I \phi[\alpha]$
Now we can interpret the truth of a WFF \( \phi \) with the respect to a model \( I \) and an assignment \( \alpha : \mathcal{V} \to |I| \).

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- If \( \phi \) satisfies \( I \) for all variable assignments \( \alpha \), then \( \phi \) satisfies \( I \) or \( I \) is a model of \( \phi \).
The Satisfaction Relation

- $I |= \rho(t_1, \ldots, t_k)[\alpha]$ iff $(t_1[\alpha], \ldots, t_k[\alpha]) \in \rho'$
The Satisfaction Relation

- $I \models \rho(t_1, \ldots, t_k)[\alpha]$ iff $(t_1[\alpha], \ldots, t_k[\alpha]) \in \rho'$
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- $I \models \phi \land \psi[\alpha]$ iff $I \models \phi[\alpha]$ and $I \models \psi[\alpha]$
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- \( I \models \rho(t_1, \ldots, t_k)[\alpha] \) iff \((t_1[\alpha], \ldots, t_k[\alpha]) \in \rho'\)
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- \( I \models \phi \land \psi[\alpha] \) iff \( I \models \phi[\alpha] \) and \( I \models \psi[\alpha] \)
- \( I \models \phi \lor \psi[\alpha] \) iff \( I \models \phi[\alpha] \) or \( I \models \psi[\alpha] \)
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- $I \models \exists x \phi[\alpha]$ iff there is an assignment $\alpha' : \mathcal{V} \uplus \{x\} \rightarrow |I|$ that extends $\alpha$ such that $I \models \phi[\alpha']$
Examples of WFF Satisfaction

Let $I, J$ be the structures we talked about before, with $|I| = |J| = \mathbb{N}$ and:

- $\exists x \text{ even} \, (x)$
- $\text{even} \, (x) \rightarrow \text{succ} \, (x)$
- $\text{even} \, (x) \rightarrow \text{even} \, (\text{succ} \, (x))$
- $\forall x \text{ even} \, (x) \rightarrow \text{even} \, (\text{succ} \, (\text{succ} \, (x))))$
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First Order Logic Semantics

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- $\text{even}(x) \rightarrow \text{even}(\text{succ}(x))$
- $\forall x \text{ even}(x) \rightarrow \text{even}(\text{succ}(\text{succ}(x)))$
- $\neg\text{even}(\text{succ}(\text{succ}(\text{zero})))$
Valid, Satisfiable, Unsatisfiable WFF’s

- WFF $\phi$ is:
Valid, Satisfiable, Unsatisfiable WFF’s

- WFF $\phi$ is:
  - valid if $I \models \phi$ for all models $I$

Example:
- Suppose $\phi$ is $\text{isPrime}(\text{Three})$ and $\psi$ is $\neg \text{isPrime}(\text{Three})$.
- What can you say about the satisfiability of $\phi \land \psi$?
- What about $\phi \lor \psi$?
Valid, Satisfiable, Unsatisfiable WFF’s

- WFF $\phi$ is:
  - **valid** if $I \models \phi$ for all models $I$
  - **satisfiable** if there is a model $I$ such that $I \models \phi$

Suppose that $\phi$ and $\psi$ are two satisfiable formulas.

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- Say, $\phi$ is $\text{isPrime}(3)$ and $\psi$ is $\neg \text{isPrime}(3)$.

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Note that all valid WFF’s are also satisfiable.

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Which of the following sentences are satisfiable? Which of them are valid?
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- \( \exists x \ greater(x, y) \)
Examples of Valid, Satisfiable, Unsatisfiable WFF’s

Which of the following sentences are satisfiable? Which of them are valid?

- $\exists x \ greater(x, y)$
- $\exists x \ greater(succ(succ(x)))$
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- $\exists x \ greater(x, y)$
- $\exists x \ greater(succ(succ(x)))$
- $\forall x \ x = zero \lor greater(x, zero)$
- $\forall x \ \forall y \ (even?(x) \land greater(x, y)) \rightarrow even?(x)$
Examples of Valid, Satisfiable, Unsatisfiable WFF’s

Which of the following sentences are satisfiable? Which of them are valid?

- \( \exists x \ greater(x, y) \)
- \( \exists x \ greater(succ(succ(x))) \)
- \( \forall x \ x = \text{zero} \lor greater(x, \text{zero}) \)
- \( \forall x \ \forall y \ (even?(x) \land greater(x, y)) \rightarrow even?(x) \)
- \( \forall x \ \exists y \ (even?(x) \lor greater(x, y)) \land \neg even?(x) \)
Which of the following sentences are satisfiable? Which of them are valid?

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- $\forall x \ \exists y \ (even?(x) \lor greater(x, y)) \land \neg even?(x)$
- $\forall x \ \forall y \ greater(x, y) \rightarrow greater(y, x)$
First Order Logic Semantics

- WFF Satisfaction
- Interpreting the Truth of WFF's

Lecture Outline

- FOL models
  - What are FOL models?
  - Isomorphisms of models
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Let $\phi$ and $\psi$ be two WFF’s
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- $\psi$ is a *logical consequence* of $\phi$ iff all models of $\phi$ are models of $\psi$
- $\phi \models \psi$ - for all models $I$, if $I \models \phi$, then $I \models \psi$
Let $\phi$ and $\psi$ be two WFF’s

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Logical Consequence and Equivalence

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  - $\phi \models \psi$ - for all models $I$, if $I \models \phi$, then $I \models \psi$
  - Examples:
    - $\text{even?}(\text{zero}) \models \exists x \text{ even?}(x)$
Let $\phi$ and $\psi$ be two WFF’s

- $\psi$ is a *logical consequence* of $\phi$ iff all models of $\phi$ are models of $\psi$
  - $\phi \models \psi$ - for all models $I$, if $I \models \phi$, then $I \models \psi$
- Examples:
  - $\text{even?(zero)} \models \exists x \text{ even?}(x)$
  - How about $\forall x \text{ even?}(x) \models \text{even?}(\text{succ(zero)})$ ?
Let $\phi$ and $\psi$ be two WFF’s

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    - How about $\exists x \phi \models \forall x \phi$?
First Order Logic Semantics

Logical Consequence and Equivalence

Let $\phi$ and $\psi$ be two WFF's

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  - $\phi \models \psi$ - for all models $I$, if $I \models \phi$, then $I \models \psi$
  - Examples:
    - $\text{even}(\text{zero}) \models \exists x \ \text{even}(x)$
    - How about $\forall x \ \text{even}(x) \models \text{even}(\text{succ}(\text{zero}))$?
    - How about $\exists x \ \phi \models \forall x \ \phi$?
    - How about $\forall x \ \phi \models \exists x \ \phi$?
Let $\phi$ and $\psi$ be two WFF’s

- $\psi$ is a **logical consequence** of $\phi$ iff all models of $\phi$ are models of $\psi$
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  - Examples:
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    - How about $\exists x \phi \models \forall x \phi$ ?
    - How about $\forall x \phi \models \exists x \phi$ ?

- $\psi$ and $\phi$ are semantically **equivalent** iff they have exactly the same models
Let \( \phi \) and \( \psi \) be two WFF’s

- \( \psi \) is a *logical consequence* of \( \phi \) iff all models of \( \phi \) are models of \( \psi \)
  - \( \phi \models \psi \) - for all models \( I \), if \( I \models \phi \), then \( I \models \psi \)
  - Examples:
    - \( \text{even?}(\text{zero}) \models \exists x \text{ even?}(x) \)
    - How about \( \forall x \text{ even?}(x) \models \text{even?}(\text{succ}(\text{zero})) \) ?
    - How about \( \exists x \ \phi \models \forall x \ \phi \) ?
    - How about \( \forall x \ \phi \models \exists x \ \phi \) ?

- \( \psi \) and \( \phi \) are semantically *equivalent* iff they have exactly the same models
  - \( \psi \) and \( \phi \) are equivalent iff \( \phi \models \psi \) and \( \psi \models \phi \)
Let $\phi$ and $\psi$ be two WFF’s

- $\psi$ is a *logical consequence* of $\phi$ iff all models of $\phi$ are models of $\psi$.
  - $\phi \models \psi$ - for all models $I$, if $I \models \phi$, then $I \models \psi$.
  - Examples:
    - $\text{even?}(\text{zero}) \models \exists x \ \text{even?}(x)$
    - How about $\forall x \ \text{even?}(x) \models \text{even?}(\text{succ}(\text{zero}))$?
    - How about $\exists x \ \phi \models \forall x \ \phi$?
    - How about $\forall x \ \phi \models \exists x \ \phi$?

- $\psi$ and $\phi$ are semantically *equivalent* iff they have exactly the same models.
  - $\psi$ and $\phi$ are equivalent iff $\phi \models \psi$ and $\psi \models \phi$.
  - Example: $\forall x \ \text{even?}(x)$ and $\text{even?}(x)$.
Some More Examples of Equivalent WFF’s

- De Morgan’s laws:
Some More Examples of Equivalent WFF’s

- De Morgan’s laws:
  - \( \neg ( \phi \land \psi ) \) and \( \neg \phi \lor \neg \phi \)
Some More Examples of Equivalent WFF’s

- De Morgan’s laws:
  - \( \neg (\phi \land \psi) \) and \( \neg \phi \lor \neg \phi \)
  - \( \neg (\phi \lor \psi) \) and \( \neg \phi \land \neg \phi \)
Some More Examples of Equivalent WFF’s

- De Morgan’s laws:
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- What can you say about the following?
Some More Examples of Equivalent WFF’s

- De Morgan’s laws:
  - $\neg(\phi \land \psi)$ and $\neg \phi \lor \neg \phi$
  - $\neg(\phi \lor \psi)$ and $\neg \phi \land \neg \phi$

- What can you say about the following?
  - $\neg \neg \phi$ and $\phi$
Some More Examples of Equivalent WFF’s

- De Morgan’s laws:
  - \( \neg (\phi \land \psi) \) and \( \neg \phi \lor \neg \psi \)
  - \( \neg (\phi \lor \psi) \) and \( \neg \phi \land \neg \psi \)

- What can you say about the following?
  - \( \neg \neg \phi \) and \( \phi \)
  - \( (\exists x \ \phi \land (\exists x \ \psi)) \) and \( (\exists x \ \phi) \land (\exists x \ \psi) \)
Some More Examples of Equivalent WFF’s

- De Morgan’s laws:
  - \( \neg (\phi \land \psi) \) and \( \neg \phi \lor \neg \phi \)
  - \( \neg (\phi \lor \psi) \) and \( \neg \phi \land \neg \phi \)

- What can you say about the following?
  - \( \neg \neg \phi \) and \( \phi \)
  - \( (\exists x \, \phi \land (\exists x \, \psi)) \) and \( (\exists x \, \phi) \land (\exists x \, \psi) \)
  - \( (\exists x \, \phi \land (\exists y \, \psi)) \) and \( (\exists x \, \phi) \land (\exists y \, \psi) \)
Some More Examples of Equivalent WFF’s

- De Morgan’s laws:
  - \( \neg(\phi \land \psi) \) and \( \neg\phi \lor \neg\phi \)
  - \( \neg(\phi \lor \psi) \) and \( \neg\phi \land \neg\phi \)

- What can you say about the following?
  - \( \neg\neg\phi \) and \( \phi \)
  - \( (\exists x \ \phi \land (\exists x \ \psi)) \) and \( (\exists x \ \phi) \land (\exists x \ \psi) \)
  - \( (\exists x \ \phi \land (\exists y \ \psi)) \) and \( (\exists x \ \phi) \land (\exists y \ \psi) \)
  - \( \exists x \ \phi \land \exists x \ \psi \) and \( \exists x \ \phi \land \psi \)
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What can you say about the following?
- \( \neg\neg\phi \) and \( \phi \)
- \( (\exists x \phi \land (\exists x \psi)) \) and \( (\exists x \phi) \land (\exists x \psi) \)
- \( (\exists x \phi \land (\exists y \psi)) \) and \( (\exists x \phi) \land (\exists y \psi) \)
- \( \exists x \phi \land \exists x \psi \) and \( \exists x \phi \land \psi \)
- \( \neg\forall x \phi \) and \( \exists x \neg\phi \)

\( \neg\exists x \phi \) and \( \forall x \neg\phi \)

\( \exists x \forall y \phi \) and \( \forall y \exists x \phi \)
Some More Examples of Equivalent WFF’s

- De Morgan’s laws:
  - \( \neg(\phi \land \psi) \) and \( \neg\phi \lor \neg\phi \)
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  - \( (\exists x \, \phi \land (\exists y \, \psi)) \) and \( (\exists x \, \phi) \land (\exists y \, \psi) \)
  - \( \exists x \, \phi \land \exists x \, \psi \) and \( \exists x \, \phi \land \psi \)
  - \( \neg\forall x \, \phi \) and \( \exists x \, \neg\phi \)
  - \( \neg\exists x \, \phi \) and \( \forall x \, \neg\phi \)
Some More Examples of Equivalent WFF’s

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  - \( \exists x \phi \land \exists x \psi \) and \( \exists x \phi \land \psi \)
  - \( \neg\forall x \phi \) and \( \exists x \neg\phi \)
  - \( \neg\exists x \phi \) and \( \forall x \neg\phi \)
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