So Far

• Search as a general paradigm for AI
  – Useful tool
  – Finding path to solution
  – Decision making in adversarial situations

• State space
  – Defined as a set of states
  – A generating function of children of a node

• How do we generate the state space?
State Space is the Final Frontier…

• Propositional Logic:
  – Propositional symbols: $x_1, x_2, \ldots, x_n$
  \[ \rightarrow <v_1, \ldots, v_n> \text{ is a state} \]
  \[ (v_i \text{ is a value (T or F) for } x_i) \]

• Generate state space by defining propositions

• How do we represent information about this state space?
Your Previous Classes

• Propositional logic:
  – **Language**: prop. symbols, connectives, formulas
  – **Semantics**: truth assignments, models, truth tables (the set of models of a formula)
Today and Thursday

- First-Order Logic (FOL)
- Language of FOL
- Semantics of FOL
- Deduction theorem for FOL
- Soundness, completeness, and incompleteness theorems
Examples

• Tarski’s World
• http://www.bu.edu/linguistics/UG/course/lx502-s04/local/tarski.html
First-order logic

• First-order logic (FOL) models the world in terms of
  – Objects, which are things with individual identities
  – Properties of objects that may or may not distinguish them from other objects
  – Relations that hold among sets of objects
  – Functions, which are a subset of relations where there is only one “value” for any given “input”

• Examples:
  – Objects: Students, lectures, companies, cars ...
  – Relations: Brother-of, bigger-than, outside, part-of, has-color, occurs-after, owns, visits, precedes, ...
  – Properties: blue, oval, even, large, ...
  – Functions: father-of, best-friend, second-half, one-more-than ...
Vocabulary of a FOL Theory

• Constant symbols, which represent individuals in the world
  – Mary
  – 3
  – Green

• Function symbols, which map individuals to individuals
  – father-of(Mary) = John
  – color-of(Sky) = Blue

• Predicate symbols, which map individuals to truth values
  – greater(5,3)
  – green(Grass)
  – color(Grass, Green)
FOL Syntax

• Variable symbols for objects in the world
  – E.g., x, y, foo

• Connectives
  – Same as in PL: not (~), and (^), or (v), implies (=>), if and only if (<=>

• Quantifiers
  – Universal \( \forall x \) or (Ax)
  – Existential \( \exists x \) or (Ex)
Quantifiers

• Universal quantification
  – \((\forall x)P(x)\) means that \(P\) holds for all values of \(x\) in the domain associated with that variable
  – E.g., \((\forall x)\) dolphin\((x)\) \(\Rightarrow\) mammal\((x)\)

• Existential quantification
  – \((\exists x)P(x)\) means that \(P\) holds for some value of \(x\) in the domain associated with that variable
  – E.g., \((\exists x)\) mammal\((x)\) \(\land\) lays-eggs\((x)\)
  – Permits one to make a statement about some object without naming it
Sentences and WFFs

• A term (denoting a real-world individual) is a constant symbol, a variable symbol, or an n-place function of n terms.
  – $x$ and $f(x_1, ..., x_n)$ are terms, where each $x_i$ is a term.
  – A term with no variables is a ground term

• An atomic formula (which has value true or false) is either
  – an n-place predicate of n terms, or, term = term

• A formula is
  – an atomic formula
  – $\neg P(x)$, $P(x) \lor Q(y)$, $P(x) \land Q(y)$, $P(x) \Rightarrow Q(y)$, $P(x) \Leftrightarrow Q(y)$ where $P(x)$ and $Q(y)$ are formulas
  – if $P(x)$ is a formula and $x$ is a variable, then $(\forall x)P(x)$ and $(\exists x)P(x)$ are formulas

• A sentence is a formula containing no “free” variables. i.e., all variables are “bound” by universal or existential quantifiers.
  – $(\forall x)R(x,y)$ has $x$ bound as a universally quantified variable, but $y$ is free.

• Well-formed formula (wff) is a formula built as described above
Quantifiers

- Universal quantifiers are often used with “implies” to form “rules”:
  - \((\forall x)\) student\(x\) => smart\(x\) means “All students are smart”

- Universal quantification is rarely used to make blanket statements about every individual in the world:
  - \((\forall x)\) student\(x\) ^ smart\(x\) means “Everyone in the world is a student and is smart”

- Existential quantifiers are usually used with “and” to specify a list of properties about an individual:
  - \((\exists x)\) student\(x\) ^ smart\(x\) means “There is a student who is smart”

- A common mistake is to represent this English sentence as the FOL sentence:
  - \((\exists x)\) student\(x\) => smart\(x\)
    
    *What’s the problem?*
Quantifier Scope

• Switching the order of universal quantifiers does not change the meaning:
  \[(\forall x)(\forall y)P(x,y) \iff (\forall y)(\forall x)P(x,y)\]

• Similarly, you can switch the order of existential quantifiers:
  \[(\exists x)(\exists y)P(x,y) \iff (\exists y)(\exists x)P(x,y)\]

• Switching the order of universals and existential does change meaning:
  – Everyone likes someone: \((\forall x)(\exists y)\text{likes}(x,y)\)
  – Someone is liked by everyone: \((\exists y)(\forall x)\text{likes}(x,y)\)
Connections between All and Exists

- We can relate sentences involving $\forall$ and $\exists$ using De Morgan’s laws:
  - $(\forall x) \sim P(x) \iff \sim(\exists x) P(x)$
  - $\sim(\forall x)P(x) \iff (\exists x) \sim P(x)$
  - $(\forall x) P(x) \iff \sim (\exists x) \sim P(x)$
  - $(\exists x) P(x) \iff \sim(\forall x) \sim P(x)$
Translating English to FOL

- Every gardener likes the sun.
  - $(\forall x)\ gardener(x) \Rightarrow likes(x, Sun)$
- All purple mushrooms are poisonous.
  - $(\forall x)\ (mushroom(x) \wedge purple(x)) \Rightarrow poisonous(x)$
- No purple mushroom is poisonous.
  - $\neg (\exists x)\ purple(x) \wedge mushroom(x) \wedge poisonous(x)$
  - $(\forall x)\ (mushroom(x) \wedge purple(x)) \Rightarrow \neg poisonous(x)$
- There are exactly two purple mushrooms.
  - $(\exists x)(\exists y)\ mushroom(x) \wedge purple(x) \wedge mushroom(y) \wedge purple(y) \wedge \neg (x=y) \wedge (Az)$
  - $(mushroom(z) \wedge purple(z)) \Rightarrow ((x=z) \vee (y=z))$
- Harry is not tall.
  - $\neg tall(Harry)$
- X is above Y if X is on directly on top of Y or there is a pile of one or more other objects directly on top of one another starting with X and ending with Y.
  - $(\forall x)(\forall y)\ above(x,y) \iff (on(x,y) \vee (\exists z)\ (on(x,z) \wedge above(z,y)))$
- You can fool some of the people all of the time.
  - $(\exists x)\ (\forall t)\ can-fool(x,t)$
  - $(\exists x)\ (person(x) \wedge ((\forall t)\ (time(t) \Rightarrow can-fool(x,t))))$
- You can fool all of the people some of the time.
  - $(\forall x)(\exists t)\ can-fool(x,t)$
  - $(\forall x)\ (person(x) \Rightarrow ((\exists t)\ time(t) \wedge can-fool(x,t)))$
Axioms, definitions and theorems

• Axioms are facts and rules that attempt to capture all of the (important) facts and concepts about a domain; axioms can be used to prove theorems
  – Mathematicians don’t want any unnecessary (dependent) axioms – ones that can be derived from other axioms
  – Dependent axioms can make reasoning faster, however
  – Choosing a good set of axioms for a domain is a kind of design problem

• A definition of a predicate is of the form “p(X) <=> …” and can be decomposed into two parts
  – Necessary description: “p(x) => …”
  – Sufficient description “p(x) <= …”
  – Some concepts don’t have complete definitions (e.g., person(x))
Axioms for Set Theory in FOL

1. The only sets are the empty set and those made by adjoining something to a set:
   \[ \forall s \text{ set}(s) \iff (s=\text{EmptySet}) \vee (\exists x,r \text{ Set}(r) \land s=\text{Adjoin}(s,r)) \]

2. The empty set has no elements adjoined to it:
   \[ \neg \exists x,s \text{ Adjoin}(x,s)=\text{EmptySet} \]

3. Adjoining an element already in the set has no effect:
   \[ \forall x,s \text{ Member}(x,s) \iff s=\text{Adjoin}(x,s) \]

4. The only members of a set are the elements that were adjoined into it:
   \[ \forall x,s \text{ Member}(x,s) \iff \exists y,r (s=\text{Adjoin}(y,r) \land (x=y \lor \text{Member}(x,r))) \]

5. A set is a subset of another iff all of the 1st set’s members are members of the 2nd:
   \[ \forall s,r \text{ Subset}(s,r) \iff (\forall x \text{ Member}(x,s) \implies \text{Member}(x,r)) \]

6. Two sets are equal iff each is a subset of the other:
   \[ \forall s,r (s=r) \iff (\text{subset}(s,r) \land \text{subset}(r,s)) \]

7. Intersection
   \[ \forall x,s1,s2 \text{ member}(X,\text{intersection}(S1,S2)) \iff \text{member}(X,s1) \land \text{member}(X,s2) \]

8. Union
   \[ \exists x,s1,s2 \text{ member}(X,\text{union}(s1,s2)) \iff \text{member}(X,s1) \lor \text{member}(X,s2) \]
First-Order Theories

• Signature L:
  – Function symbols \( f(x,y) \)
  – Predicate (relation) symbols \( P(x,A) \)
  – Constant symbols \( A,B,C,… \)

• FOL language: quantification over objects

\[
\forall w \exists m \left( \text{woman} \ (w) \land \text{man} \ (m) \land \text{loves} \ (m, w) \right)
\]
\[
\forall w \exists m \left( \text{woman} \ (w) \rightarrow \text{man} \ (m) \land \text{loves} \ (m, w) \right)
\]
\[
\forall w \left( \text{woman} \ (w) \rightarrow \exists m \left( \text{man} \ (m) \land \text{loves} \ (m, w) \right) \right)
\]
Well Founded Formulae

• Generate formulae from signature together with variables, connectives, quantifiers, and parentheses
  – Terms: variables, constants, f(a1, …)
  – Atomic formula: p(a1, a2, …)
  – Formula (negation, disjunction, existential)

• Not wffs…
Knowledge Representation in FOL

- Spatial relationships
- Temporal relationships
- Choice of concepts, objects, functions
- Elaboration tolerance
Model Theory

• Structure/Interpretation: \(<U,I>\)
  – \(U\) = Universe of elements
  – \(I\) = Mapping of
    • Constant symbols to elements in \(U\)
    • Predicate symbols to relations over \(U\)
    • Function symbols to functions over \(U\)

• \(M \models T\) - \(M\) satisfies \(T\)
  – \(T\) is a theory, i.e., a set of FOL sentences in language \(\mathcal{L}\) for which \(M\) is an interpretation
Logical Entailment

\[ \alpha = \forall w \exists m(\text{woman } (w) \land \text{man } (m) \land \text{loves } (m, w)) \]
\[ \beta = \forall w \exists m(\text{woman } (w) \rightarrow \text{man } (m) \land \text{loves } (m, w)) \]
\[ \gamma = \forall w(\text{woman } (w) \rightarrow \exists m(\text{man } (m) \land \text{loves } (m, w))) \]

\[ \models \beta \rightarrow \alpha \quad ? \]
\[ \models \gamma \Rightarrow \beta \quad ? \]

\( L = \{\text{man, woman, loves}\} \)
\( M_1 = <U_1,I_1> \)
\( U_1 = \{\text{Sue, Kim, Pat}\} \)
\( I_1[\text{man}] = \{\text{Pat}\} \)
\( I_1[\text{woman}] = \{\text{Sue, Kim}\} \)
\( I_1[\text{loves}] = \{<\text{Pat, Kim}>, <\text{Pat, Sue}>\} \)

\( M_1 \models \beta \)
\( M_1 \not\models \alpha \)
Model Checking

• Given an interpretation, how do we decide if it satisfies a formula (i.e., gives it a truth value of TRUE)?
• What happens when the formula is not closed?
Proof and Theorems

• Axiom systems and rule systems
  – Propositional axioms: \(-A \lor A\)
  – Substitution axioms: \(A_x[a] \rightarrow \exists xA\)
  – Identity axioms: \(x=x\)
  – Equality axioms:
    \[x_1=y_1 \rightarrow \ldots \rightarrow x_n=y_n \rightarrow f(x_1,\ldots,x_n)=f(y_1,\ldots,y_n)\]
    \[x_1=y_1 \rightarrow \ldots \rightarrow x_n=y_n \rightarrow p(x_1,\ldots,x_n)=p(y_1,\ldots,y_n)\]

• Rules:...
Proof and Theorems

• Axiom systems and rule systems
  – ...

• Rules:
  – Expansion Rule: Infer BvA from A
  – Contraction Rule: Infer A from AvA
  – Associative Rule: Infer (AvB)vC from Av(BvC)
  – Cut Rule: Infer BvC from AvB and –AvC
  – \(\exists\)-introduction rule: If x not free in B, then infer \(\existsxA\rightarrow B\) from \(A\rightarrow B\)
Proofs and Theorems

- Nonlogical axioms: Axioms made for a certain theory
- Proofs – what are they?
- Completeness, Soundness (of an inference procedure)
- Incompleteness of FOL (as a language) for some intended models
Clausal Form

• Every FOL formula is *consistency-equivalent* to conjunction of F.O. clauses.

\[ \forall w (\text{woman} (w) \rightarrow \exists m (\text{man} (m) \land \text{loves} (m, w))) \]

• First-order clause
  – Only universal quantifiers (which are implicit)
  – Disjunction of literals (atoms or their negation)

\[ \neg \text{woman} (w) \lor \text{man} (SKm (w)) \]
\[ \neg \text{woman} (v) \lor \text{loves} (SKm (v), v) \]
Conversion to Clausal Form

1. “→” replaced by “¬”, ”∨”

$$\forall w (w \text{oman} (w) \rightarrow \exists m (\text{man} (m) \land \text{loves} (m, w)))$$

2. Negations in front of atoms

$$\forall w (\neg w \text{oman} (w) \land \neg \exists m (\text{man} (m) \land \text{loves} (m, w)))$$

3. Standardize variables (unique vars.)

$$\forall w (w \text{oman} (w) \rightarrow \exists m (\text{man} (m) \land \text{loves} (m, w)))$$

4. Eliminate existentials (Skolemization)

$$\forall w (w \text{oman} (w) \rightarrow \exists m (\text{SKm (m)} \land \text{loves (m, w)}))$$

5. Drop all universal quantifiers

$$\forall w (\neg w \text{oman} (w) \land \neg \exists m (\text{man} (m) \land \text{loves} (m, w)))$$

6. Move disjunctions in (put into CNF)

$$\forall w (\neg \text{oman} (w) \lor \neg \text{man (SKm (w))} \land \text{loves (SKm (w), w))}$$

7. Rename vars. (standardize vars. apart)

\[
\begin{aligned}
&\neg \text{oman (w) } \lor \text{man (SKm (w))} \land \\
&\left( \neg \text{oman (w)} \lor \text{loves (SKm ((w), )w}))\right) \\
&\left( \neg \text{oman (w)} \lor \text{loves (SKm ((w), )v}))\right)
\end{aligned}
\]
Next Time

• Reasoning procedure for FOL
  – Proving entailment using Resolution

• Application du jour: Temporal Reasoning

• Applications we will *not* touch
  – Spatial reasoning, formal verification, mathematics, planning, NLP, …